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Research article

A Novel Extension of the Fréchet Distribution: Statistical Properties and Application to Groundwater Pollutant Concentrations

Ahmad Abubakar Suleiman [1,2*], *Hanita Daud* [1], *Mahmod Othman* [3], *Narinderjit Singh Sawaran Singh* [4] *Aliyu Ismail Ishaq* [5], *Rajalingam Sokkalingam* [1], *Abdullah Husin* [3]

[1] *Fundamental and Applied Sciences Department, Universiti Teknologi PETRONAS 32610 Seri Iskandar, Perak Darul Ridzuan, Malaysia;*
[2] *Department of Statistics, Aliko Dangote University of Science and Technology, Wudil 713281, Nigeria;*
[3] *Department of Information System, Universitas Islam Indragiri, Tembilahan, Indonesia;*
[4] *Faculty of Data Science and Information Technology, INTI International University, Persiaran Perdana BBN Putra Nilai, 71800 Nilai, Negeri Sembilan, Malaysia;*
[5] *Department of Statistics, Ahmadu Bello University, 810107 Zaria, Nigeria;*
email: [1,*] *ahmad_22000579@utp.edu.my, hanita_daud@utp.edu.my, raja.sokkalingam@utp.edu.my,*
[2] *ahmadabubakar31@kustwudil.edu.ng,* [3] *mahmod.othman@unisi.ac.id, abdialam@gmail.com*
[4] *narinderjits.sawaran@newinti.edu.my,* [5] *aiishaq@abu.edu.ng*
* *Correspondence : ahmad_22000579@utp.edu.my (A.A.S.)*

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Correspondence:
Department of Statistics, Faculty of Computing and Mathematical Sciences, Aliko Dangote University of Science and Technology, Wudil 713281, Nigeria.

ABSTRACT

In this work, we propose and study a novel generalization of the Fréchet distribution called the odd beta prime Fréchet (OBPF) distribution. This distribution was an extension of the Fréchet distribution by applying the odd beta prime generalized family of distributions. The proposed model can be expressed as a linear mixture of Fréchet densities. The shapes of the density function possess great flexibility. It can accommodate various hazard shapes, such as concave-convex, increasing, decreasing, and J-shaped. Some important statistical properties of the OBPF are derived, including the ordinary and incomplete moments, order statistics, and quantile function. We have used the maximum likelihood estimation method to estimate the model parameters. The application and flexibility of the new distribution are empirically proven using groundwater pollution data sets compared to other competing distributions. The new model can be used instead of existing lifetime distributions and is suitable to fit data with right-skewed and left-skewed behaviors.

1. Introduction

Groundwater hydrology studies water flow, storage, chemical transport, and related processes in the subsurface . [1] [2] Groundwater specialists examine the features of rocks and soil to get an understanding of the subsurface hydrologic processes and develop predictive models for groundwater phenomena. These features include porosity, permeability, hydraulic conductivity, specific storage, and specific yield. Because of the complexity of geologic materials, measurements of these features vary even in strata considered homogeneous because of their origin and basic properties. These measurements and those of groundwater quality can be evaluated using the laws of probability and statistics to obtain a clear description of the investigated variable that goes beyond the calculation of its mean, standard deviation, and other measures of central tendency, dispersion, and asymmetry [3] [4]. Recent research has shown how statistical distributions can be used to model data in applied sciences, especially environmental science. Statisticians often explore new statistical models to suit data sets in diverse domains. Statistical models are very useful in describing and predicting real phenomena. Many distributions have been widely used for data modeling in several domains during the last few decades. Recent developments focus on defining new families that extend well-known distributions and, at the same time, provide great flexibility in data modeling in practice. Thus, the fitting of groundwater quality with an appropriate probability density function (pdf) is necessary for a comprehensive description of its probabilistic features.

The Fréchet distribution was introduced by [5] to model extreme events, including annual maximum one-day rainfalls, wind speeds, sea waves, earthquakes, floods, and river discharges. The inverse Rayleigh and inverse exponential distributions can be considered special cases of the Fréchet distribution. This distribution is widely used in extreme value theory. The Fréchet distribution, also known as the extreme value distribution (EVD) Type II, is used to model maximum values in a data set. It is one of four EVDs in common use. The other three are the Gumbel distribution, the Weibull distribution, and the generalized extreme value distribution. This distribution is used to model a wide range of phenomena like horse racing, queues in banks, and human lifespans. Some further information about the Fréchet distribution, including its applications, is available in [6] [7] [8] [9] [10] and the reference therein. The Fréchet has the following cumulative distribution (cdf):

$$F\left(x\right)=e^{-\left(\frac{\beta}{x}\right)^{\theta}}, \quad x>0; \quad \theta,\beta>0. \tag{1}$$

The corresponding probability distribution function (pdf) is defined as

$$f\left(x\right)=\theta^{\theta}x^{-\theta-1}e^{-\left(\frac{\beta}{x}\right)^{\theta}}, \quad x>0; \quad \theta,\beta>0, \tag{2}$$

where θ and β are scale and shape parameters, respectively.

However, it is evident from the statistical literature that the convention probability distributions such as the Fréchet cannot adequately suit every kind of real-life phenomenon [11] [12][13][14]. Therefore, it is imperative to propose new distributions that can provide better flexibility in modeling real data sets, which can be done by extending traditional distributions by introducing new parameters, or by generating a new family of distributions [15]. During the last few decades, numerous generators have been developed by modifying several important statistical models that have proven effective. These generated families of distributions have been widely utilized for modeling and evaluating lifetime data in a wide variety of applied sciences, including hydrology, actuarial sciences, engineering, insurance, demographics, reliability, economics, biology, finance, medicine, and machine learning, to mention a few. However, there are numerous real-world events that do not fit any of the standard statistical models.

The statistical literature contains many generalizations of the Fréchet distribution. For example, the beta Fréchet [16], the Weibull Fréchet distribution [17], the odd Lomax Fréchet distribution [18], the odd Lindley Fréchet distribution [19], the Gompertz Fréchet distribution [20], the novel alpha power Fréchet [7], the generalized odd log-logistic Fréchet distribution [21], the odd Chen Frécht distribution [22] , and the Kumaraswamy power Fréchet distribution [23]. This paper is, however, interested in extending the Fréchet distribution using the odd beta prime family of distributions proposed by [24]. This family has been used generalized the exponential distribution by [25] and the Weibull distribution by [26]. One of the motivations for the present study is the ability of the beta prime family of distributions to perform creditably better than other well-known families of distributions.

It is worthy of note that other generalized families of distributions exist in the literature, examples of which include the generalized odd half-Cauchy by [27], the odd-Burr generalized by [28], the extended odd Fréchet-G by [12], the generalized odd Weibull-G by [29], the generalized odd gamma-G by [30], the modified odd Weibull by [31] , the odd Dagum by [32].the Zubair-G by [33], the truncated Burr X-G by [34] , the alpha power Marshall-Olkin-G by [35], the Maxwell-G by [36] the type II power Topp-Leone-G by [37], the odd log-logistic Burr-by [38], the truncated Cauchy power Weibull-G by [39], the type II half-logistic odd Fréchet by [40], the transmuted odd Fréchet-G b [41], the odd Perks-G by [42], the new truncated muth-G by [43], the odd F-Weibull distribution [44], and several other families. The odd beta prime family of distribution is well preferred in this research because it has not been widely explored despite its potential. This research is aimed at extending the Fréchet distribution in order to increase its potential in modeling groundwater pollutants and could be used for extreme events in hydrology, reliability studies, finance, and more. Therefore, In the present study, a novel model referred to as the odd beta prime Fréchet (OBPF) distribution is proposed using the odd beta prime family of distribution via the T-X approach. It is a generalization of the Fréchet distribution. We have also explored some of its statistical properties. Figure 1 shows the graphical summary of the study.

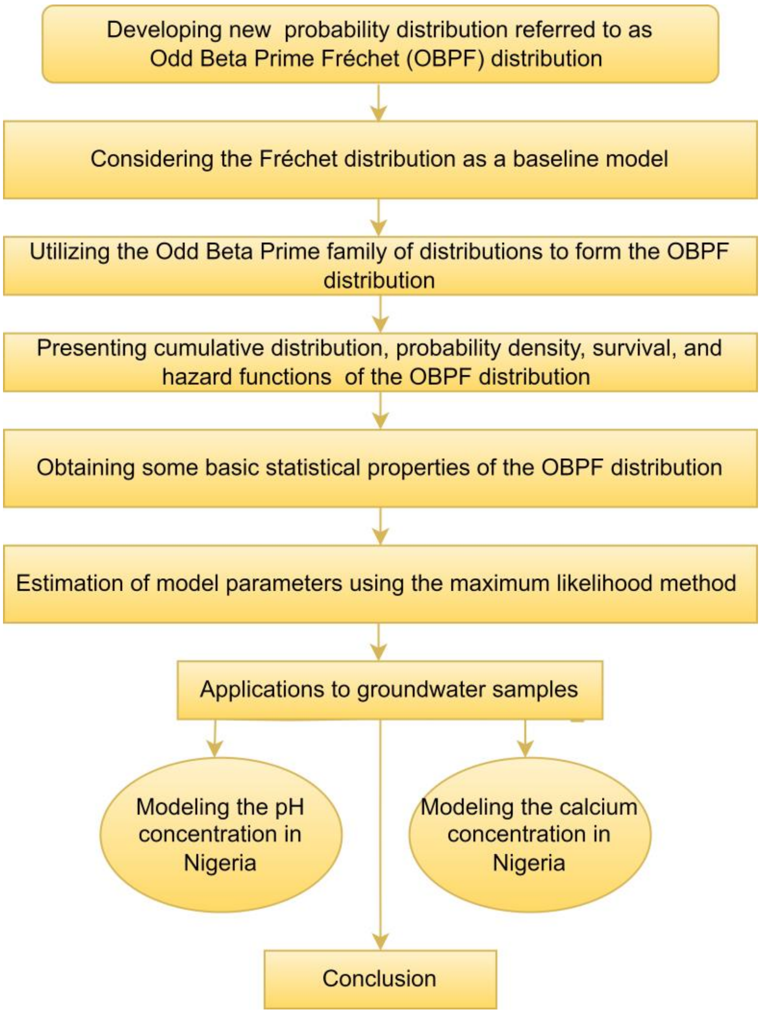

Figure 1 Graphical summary of the study.

The proposed OBPF model is motivated by some special features as follows:

i. The OBPF model improves the flexibility of the classical Fréchet distribution.

ii. The OBPF distribution provides left-skewed, right-skewed, S-shaped, and reversed-J densities and concave-concex, decreasing, increasing, and J-shaped hazard functions.

iii. The OBPF model has been employed to model a asymmetric groundwater quality data sets.

iv. The OBPF model offers better fits compared to other competitive models.

Herein, this article is outlined as follows: In Section 2, the new OBPF distribution is introduced, along with the derivation of its validity test as well as mixture representations of the OBPF distribution. Various statistical properties of the proposed distributions are obtained in Section 3. The MLE for the model parameters is discussed in Section 4. Section 5 presents applications of the proposed OBPF distribution, illustrated by means of two groundwater data sets. The concluding remarks are provided in Section 6.

2. Development of the New Odd Beta Prime Fréchet distribution

2.1. The Definition of the OBPF Distribution

The cdf of the odd beta prime generalized family of distribution is given by

$$F(x) = \frac{B_{\frac{G(x,\varepsilon)}{1-G(x,\varepsilon)}}(\alpha,\lambda)}{B(\alpha,\lambda)} \quad ; \quad x \in \mathbb{R}, \tag{3}$$

and the corresponding pdf is given by

$$f\left(x\right)=\frac{g(x,\varepsilon)}{B\left(\alpha,\lambda\right)\left\{1-G(x,\varepsilon)\right\}^{2}}\frac{\left\{\dfrac{G(x,\varepsilon)}{1-G(x,\varepsilon)}\right\}^{\alpha-1}}{\left\{1+\left(\dfrac{G(x,\varepsilon)}{1-G(x,\varepsilon)}\right)\right\}^{\alpha+\lambda}};\quad x\in\mathbb{R},\tag{4}$$

where α and λ are extra shape parameters, $G(x,\varepsilon)$ is cdf of any baseline distribution with parameter ε, $g(x,\varepsilon)$ is the corresponding pdf, $B\left(\alpha,\lambda\right)$ and $B_{x}\left(\alpha,\lambda\right)$ are the beta and incomplete beta functions with shape parameters $\alpha>0$ and $\lambda>0$, respectively. These shape parameters are to introduce/induce skewness into the parent distribution, and this, however, increased the ability of the proposed distribution to model real life events.

As a result, by substituting the expression in (1) into (3), the cdf of the OBPF distribution is as follows:

$$F\left(x\right)=\frac{B_{\frac{e^{-\left(\frac{\beta}{x}\right)^{\theta}}}{1-e^{-\left(\frac{\beta}{x}\right)^{\theta}}}}\left(\alpha,\lambda\right)}{B\left(\alpha,\lambda\right)}\;;\quad x>0,\tag{5}$$

The corresponding pdf of the OBPF distribution is given by

$$f\left(x\right)=\frac{\left\{\theta\beta^{\theta}x^{-\theta-1}e^{-\left(\frac{\beta}{x}\right)^{\theta}}\right\}}{B\left(\alpha,\lambda\right)\left\{1-e^{-\left(\frac{\beta}{x}\right)^{\theta}}\right\}^{2}}\frac{\left\{\dfrac{e^{-\left(\frac{\beta}{x}\right)^{\theta}}}{1-e^{-\left(\frac{\beta}{x}\right)^{\theta}}}\right\}^{\alpha-1}}{\left\{1+\left(\dfrac{e^{-\left(\frac{\beta}{x}\right)^{\theta}}}{1-e^{-\left(\frac{\beta}{x}\right)^{\theta}}}\right)\right\}^{\alpha+\lambda}};\quad x>0.\tag{6}$$

Equation (6) can be rewritten as

$$f\left(x\right)=\frac{\theta\beta^{\theta}x^{-\theta-1}\left\{e^{-\left(\frac{\beta}{x}\right)^{\theta}}\right\}^{\alpha}}{B\left(\alpha,\lambda\right)\left\{1-e^{-\left(\frac{\beta}{x}\right)^{\theta}}\right\}^{1-\lambda}};\quad x>0.\tag{7}$$

Which is the pdf of OBPF distribution.

To verify whether the proposed OBPF distribution is a valid probability distribution, the pdf given in (7) must satisfy $\int_{-\infty}^{\infty}f(x)dx=1$, where $f(x)$ is the pdf of OBPF distribution. For the sake of validation, we provide clear proof as follows:

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\theta\beta^{\theta}}{B(\alpha,\lambda)} \int_{0}^{\infty} \frac{x^{-\theta-1}\left\{e^{-\left(\frac{\beta}{x}\right)^{\theta}}\right\}^{\alpha}}{\left\{1-e^{-\left(\frac{\beta}{x}\right)^{\theta}}\right\}^{1-\lambda}} dx. \tag{8}$$

Let $m = e^{-\left(\frac{\beta}{x}\right)^{\theta}}$ so that $dx = \frac{x^{\theta+1}}{\theta\beta^{\theta} e^{-\left(\frac{\beta}{x}\right)^{\theta}}} dm.$ (9)

Putting (9) into (8), we have

$$f(x) = \frac{\theta\beta^{\theta}}{B(\alpha,\lambda)} \int_{0}^{1} \frac{x^{-\theta-1} m^{\alpha}}{(1-m)^{1-\lambda}} \frac{x^{\theta+1}}{\theta\beta^{\theta} m} dm,$$

$$= \frac{1}{B(\alpha,\lambda)} \int_{0}^{1} m^{\alpha-1}(1-m)^{\lambda-1} dm,$$

$$= \frac{1}{B(\alpha,\lambda)}.B(\alpha,\lambda) = 1. \tag{10}$$

Hence, the OBPF is a valid probability distribution.
The survival function for OBPF distribution is

$$\mathbb{S}(x) = 1 - F(x),$$

$$= 1 - \frac{B_{\frac{e^{-\left(\frac{\beta}{x}\right)^{\theta}}}{1-e^{-\left(\frac{\beta}{x}\right)^{\theta}}}}(\alpha,\lambda)}{B(\alpha,\lambda)} \; ; \; x > 0, \; \theta,\beta,\alpha,\lambda > 0. \tag{11}$$

The hazard function for OBPF distribution is expressed as

$$h(x) = \frac{f(x)}{\mathbb{S}(x)} = \frac{f(x)}{1-\mathbb{S}(x)},$$

$$= \frac{\dfrac{\theta\beta^{\theta} x^{-\theta-1}\left\{e^{-\left(\frac{\beta}{x}\right)^{\theta}}\right\}^{\alpha}}{B(\alpha,\lambda)\left\{1-e^{-\left(\frac{\beta}{x}\right)^{\theta}}\right\}^{1-\lambda}}}{1 - \dfrac{B_{\frac{e^{-\left(\frac{\beta}{x}\right)^{\theta}}}{1-e^{-\left(\frac{\beta}{x}\right)^{\theta}}}}(\alpha,\lambda)}{B(\alpha,\lambda)}} ; \; x > 0, \; \theta,\beta,\alpha,\lambda > 0. \tag{12}$$

Graphs of the pdf and hazard functions of the OBPF distribution at several different distribution parameters are given in Figures 2 and 3, respectively. Figure 2 shows that the new pdf can be unimodal, left-skewed, right-skewed, S-shaped, and J-shaped. Figure 3 shows that the hazard function may be concave-convex, monotonically decreasing, monotonically increasing, and J-shaped.

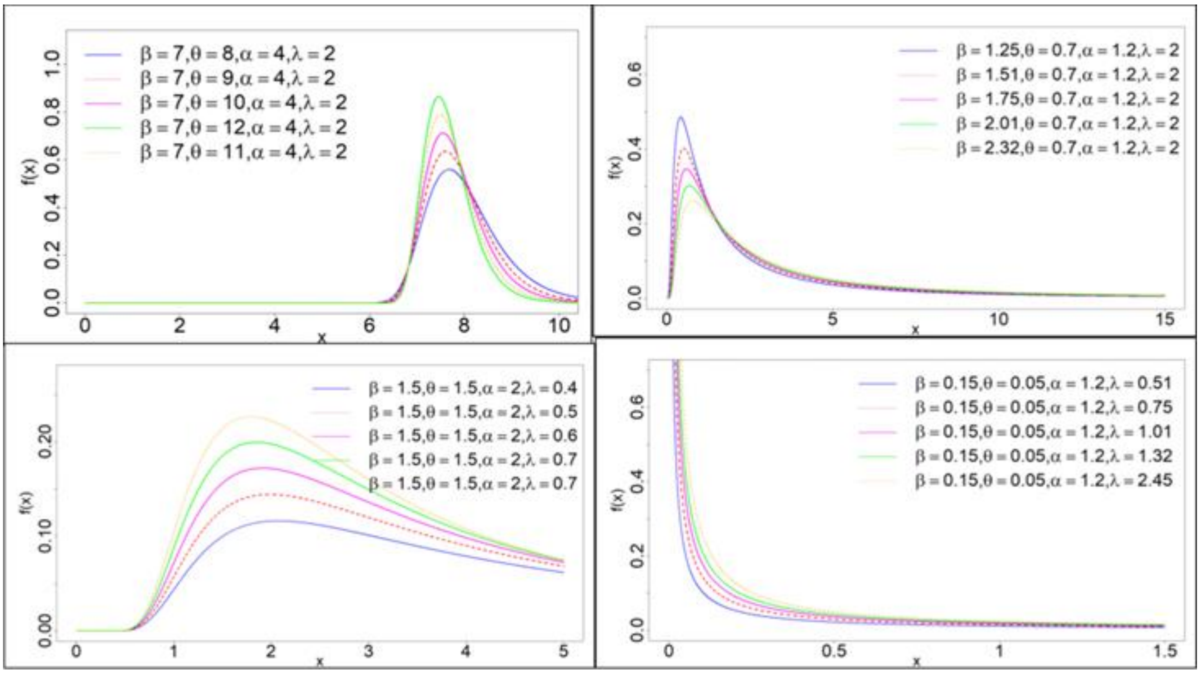

Figure 2 Density function plots of the OBPF distribution.

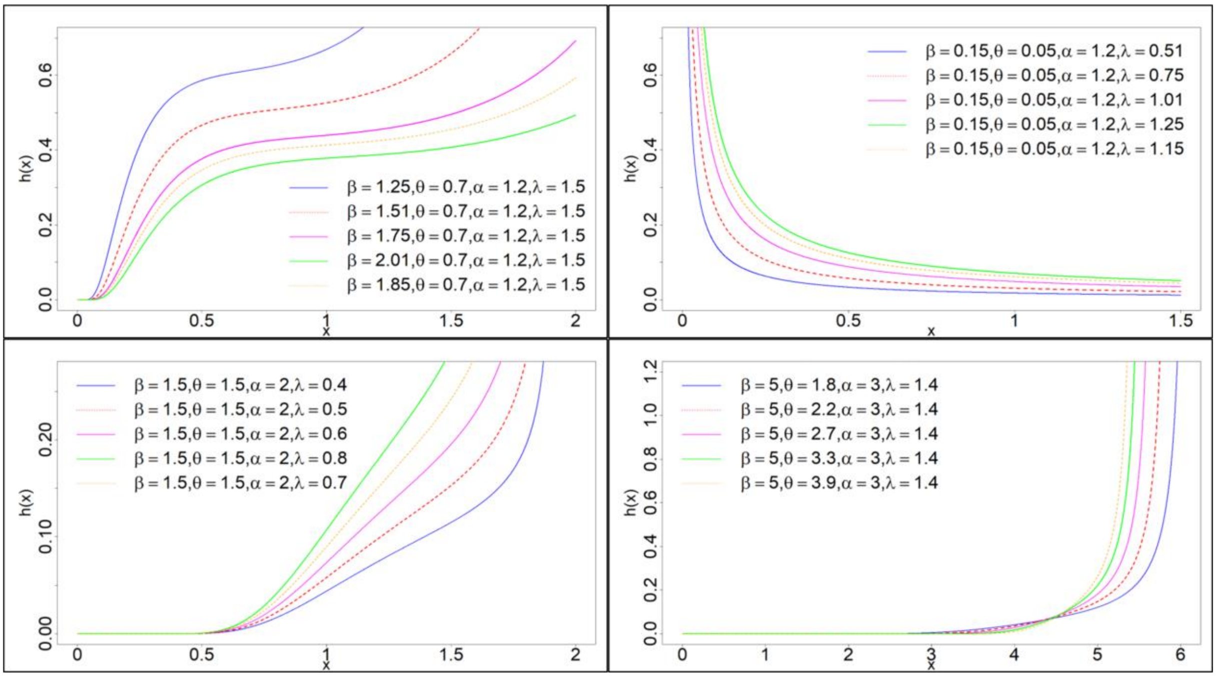

Figure 3 Hazard function plots of the OBPF distribution.

2.2. *Mixture Representations*

This subsection derives important mixture representations of the OBPF distribution defined in (7).
It is known that if $|t| < 1$ and $c > 0,$ then the power series expansion of this holds,

$$\left(1-t\right)^{m} = \sum_{i=0}^{\infty} \frac{\Gamma(m+i)}{i!\Gamma(m)} t^{i}. \tag{13}$$

Applying (13) in (7), we get the mixture representations of OBPF distribution as

$$f(x) = \frac{\theta\beta^{\theta} x^{-\theta-1} e^{-\alpha\left(\frac{\beta}{x}\right)^{\theta}}}{B(\alpha,\lambda)} \cdot \sum_{i=0}^{\infty} \frac{\Gamma(1-\lambda+i)}{i!\Gamma(1-\lambda)} e^{-i\left(\frac{\beta}{x_i}\right)^{\theta}},$$

$$= \frac{\theta\beta^{\theta} x^{-\theta-1}}{B(\alpha,\lambda)} \cdot \sum_{i=0}^{\infty} \frac{\Gamma(1-\lambda+i)}{i!\Gamma(1-\lambda)} e^{-(i+\alpha)\left(\frac{\beta}{x}\right)^{\theta}},$$

$$= x^{-\theta-1} \sum_{i=0}^{\infty} A_i e^{-(i+\alpha)\left(\frac{\beta}{x}\right)^{\theta}}, \tag{14}$$

which is the pdf of the OBPF distribution expressed in terms of mixture representations,

where $A_i = \frac{\theta\beta^{\theta}}{B(\alpha,\lambda)} \sum_{i=0}^{\infty} \frac{\Gamma(1-\lambda+i)}{i!\Gamma(1-\lambda)}$.

The expressions in (14) can further be used to obtain the moments and incomplete moments of the OBPF distribution**.**

3. Some Statistical Properties of the OBPE Distribution

Various statistical properties, including moments, incomplete moments, order statistics, and quantile function are provided in this section.

3.1. Moments

In this subsection, we derive the moments of the OBPF distribution. By the definition of moments, the moments of the OBPF distribution can be derived from mixture representations of the pdf of the OBPF distribution defined in (14) as

$$E\left(X^{r}\right) = \int_{-\infty}^{\infty} x^{r} f(x) dx,$$

$$= \sum_{i=0}^{\infty} A_i \int_{0}^{\infty} x^{r-\theta-1} e^{-(i+\alpha)\left(\frac{\beta}{x}\right)^{\theta}} dx. \tag{15}$$

Let $w = (i+\alpha)\left(\frac{\beta}{x}\right)^{\theta}$ so that $dx = -\frac{x^{\theta+1}}{\theta\beta^{\theta}(i+\alpha)} dw.$ (16)

Substituting (16) in (15), we have

$$E\left(X^{r}\right) = -\sum_{i=0}^{\infty} A_i \int_{\infty}^{0} x^{r-\theta-1} e^{-w} \frac{x^{\theta+1}}{\theta\beta^{\theta}(i+\alpha)} dw,$$

$$= \frac{\sum_{i=0}^{\infty} A_i}{\theta\beta^{\theta}(i+\alpha)} \int_{0}^{\infty} \left\{\frac{(i+\alpha)\beta^{\theta}}{w}\right\}^{\frac{r}{\theta}} e^{-w} dw,$$

$$= \frac{\sum_{i=0}^{\infty} A_i}{\theta\beta^{\theta}(i+\alpha)} \left\{(i+\alpha)\beta^{\theta}\right\}^{\frac{r}{\theta}} \int_{0}^{\infty} w^{\left(1-\frac{r}{\theta}\right)-1} e^{-w} dw,$$

$$= \frac{\sum_{i=0}^{\infty} A_i}{\theta\left\{(i+\alpha)\beta^{\theta}\right\}^{1-\frac{r}{\theta}}} \Gamma\left(1-\frac{r}{\theta}\right), \tag{17}$$

which is the moment of the OBPF distribution.

3.2. Incomplete Moments

In this subsection, we derive the incomplete moments of the OBPF distribution. By the definition, the incomplete moments of the random variable X is defined by

$$\psi_r(X) = \int_{-\infty}^{t} x^r f(x)dx,$$

$$= \sum_{i=0}^{\infty} A_i \int_0^t x^{r-\theta-1} e^{-(i+\alpha)\left(\frac{\beta}{x}\right)^{\theta}} dx,$$

$$= \frac{\sum_{i=0}^{\infty} A_i}{\theta\beta^{\theta}(i+\alpha)} \int_0^{(i+\alpha)\left(\frac{\beta}{t}\right)^{\theta}} \left\{\frac{(i+\alpha)\beta^{\theta}}{w}\right\}^{\frac{r}{\theta}} e^{-w} dw. \tag{18}$$

Since $w = (i+\alpha)\left(\frac{\beta}{x}\right)^{\theta}$, (18) can be expressed as

$$\psi_r(X) = \frac{\sum_{i=0}^{\infty} A_i}{\theta\left\{(i+\alpha)\beta^{\theta}\right\}^{1-\frac{r}{\theta}}} \int_0^{(i+\alpha)\left(\frac{\beta}{t}\right)^{\theta}} w^{\left(1-\frac{r}{\theta}\right)-1} e^{-w} dw,$$

$$\psi_r(X) = \frac{\sum_{i=0}^{\infty} A_i}{\theta\left\{(i+\alpha)\beta^{\theta}\right\}^{1-\frac{r}{\theta}}} \gamma\left(\theta, (i+\alpha)\left(\frac{\beta}{t}\right)^{\theta}\right), \tag{19}$$

which is the incomplete moments of OBPF distribution.

3.3. Order Statistics

Here, we derive the order statistics of the OBPF distribution. Suppose $X_1, X_2 ..., X_n$ be a random sample of size n from the OBPF distribution and let $X_{1,n} < X_{2,n} < ... < X_{n,n}$ is a set of random variables of n ordered, then the distribution of κ^{th} OS is expressed as

$$F_{\kappa,n}(x) = \frac{n!\, f(x)}{(\kappa-1)!(n-\kappa)!} F_{(x)}^{\kappa-1}\left[1-F(x)\right]^{n-1}$$

$$= \frac{n!\, f(x)}{(\kappa-1)!(n-\kappa)!} \sum_{\mu=0}^{\infty} (-1)^{\mu} \binom{n-\kappa}{\mu} F_{(x)}^{\kappa+\mu-1},$$

where $F(x)$ and $f(x)$ are cdf and pdf of OBPF distribution, respectively.

3.4. Quantile Function

In this subsection, we derived the quantile function of the OBPF distribution. The quantile function of OBPF distribution can be derived from an inverse function of the cdf defined in (3) as

$$F(x) = u = \frac{B_z(\alpha,\lambda)}{B(\alpha,\lambda)} = I(z;\alpha,\lambda), \tag{20}$$

where $z = \frac{e^{-\left(\frac{\beta}{x}\right)^{\theta}}}{1-e^{-\left(\frac{\beta}{x}\right)^{\theta}}}$ and u has a uniform distribution on interval 0 to 1

The quantile function of the OBPF distribution is obtained by inverting (20) as

$$\frac{e^{-\left(\frac{\beta}{x}\right)^{\theta}}}{1-e^{-\left(\frac{\beta}{x}\right)^{\theta}}}=\frac{I^{-1}(u;\alpha,\lambda)}{K}, \tag{21}$$

where $K=I^{-1}(u;\alpha,\lambda)$.

This becomes,

$$\left(e^{-\left(\frac{\beta}{x}\right)^{\theta}}\right)=K\left(1-e^{-\left(\frac{\beta}{x}\right)^{\theta}}\right). \tag{22}$$

After some simplifications, we have

$$\frac{\beta}{x}=\left\{-\log\left(\frac{K}{1+K}\right)\right\}^{\frac{1}{\theta}}. \tag{23}$$

Therefore, the quantile function of the OBPF distribution is

$$x=\beta\left\{-\log\left(\frac{K}{1+K}\right)\right\}^{-\frac{1}{\theta}}. \tag{24}$$

4. Parameter Estimation

In this section, the estimates of parameters of the OBPF distribution using the maximum likelihood method are provided. Let $X_1, X_2 ..., X_n$ be a random variable of sample size n from the OBPF distribution with parameters β, θ, α and λ, then the likelihood function of the OBPF distribution is obtained from (7) as

$$L=\left\{\frac{\theta\beta^{\theta}}{B(\alpha,\lambda)}\right\}^{n}\prod_{i=1}^{n}x_i^{-\theta-1}e^{-\alpha\left(\frac{\beta}{x_i}\right)^{\theta}}\prod_{i=1}^{n}\left\{1-e^{-\left(\frac{\beta}{x_i}\right)^{\theta}}\right\}^{-(1-\lambda)}. \tag{25}$$

The log-likelihood function of (25) denoted by l is expressed as

$$l=n\log\left\{\frac{\theta\beta^{\theta}}{B(\alpha,\lambda)}\right\}-(\theta+1)\sum_{i=1}^{n}\log(x_i)-\alpha\sum_{i=1}^{n}\left(\frac{\beta}{x_i}\right)^{\theta} -(1-\lambda)\sum_{i=1}^{n}\log\left\{1-e^{-\left(\frac{\beta}{x_i}\right)^{\theta}}\right\}. \tag{26}$$

To get the maximum likelihood estimators of the OBPF distribution, we must maximize the log-likelihood function. To achieve this purpose, we take the first partial derivative of (26) with respect to parameters β, θ, α and λ, then we obtain

$$\frac{\partial L}{\partial\beta}=\frac{n\theta}{\beta}-\frac{\alpha\theta}{\beta}\sum_{i=1}^{n}\left(\frac{\beta}{x_i}\right)^{\theta}-(1-\lambda)\sum_{i=1}^{n}\left\{\frac{1}{1-e^{-\left(\frac{\beta}{x_i}\right)^{\theta}}}\right\}\left\{\frac{\theta}{\beta}\left(\frac{\beta}{x_i}\right)^{\theta}e^{-\left(\frac{\beta}{x_i}\right)^{\theta}}\right\},$$

$$=\frac{n\theta}{\beta}-\frac{\alpha\theta}{\beta}\sum_{i=1}^{n}\left(\frac{\beta}{x_i}\right)^{\theta}-\frac{(1-\lambda)\theta}{\beta}\sum_{i=1}^{n}\left\{\frac{\left(\frac{\beta}{x_i}\right)^{\theta}e^{-\left(\frac{\beta}{x_i}\right)^{\theta}}}{1-e^{-\left(\frac{\beta}{x_i}\right)^{\theta}}}\right\}. \tag{27}$$

$$\frac{\partial L}{\partial \theta}=\frac{n}{\theta}+n\log\beta-\sum_{i=1}^{n}\log\left(x_i\right)-\alpha\sum_{i=1}^{n}\left(\frac{\beta}{x_i}\right)^{\theta}\log\left(\frac{\beta}{x_i}\right)$$
$$-\left(1-\lambda\right)\sum_{i=1}^{n}\left\{\frac{1}{1-e^{-\left(\frac{\beta}{x_i}\right)^{\theta}}}\right\}.\left\{\left(\frac{\beta}{x_i}\right)^{\theta}\log\left(\frac{\beta}{x_i}\right)e^{-\left(\frac{\beta}{x_i}\right)^{\theta}}\right\},$$
$$=\frac{n}{\theta}+n\log\beta-\sum_{i=1}^{n}\log\left(x_i\right)-\alpha\sum_{i=1}^{n}\left(\frac{\beta}{x_i}\right)^{\theta}\log\left(\frac{\beta}{x_i}\right)$$
$$-\left(1-\lambda\right)\sum_{i=1}^{n}\left\{\frac{\left(\frac{\beta}{x_i}\right)^{\theta}\log\left(\frac{\beta}{x_i}\right)e^{-\left(\frac{\beta}{x_i}\right)^{\theta}}}{1-e^{-\left(\frac{\beta}{x_i}\right)^{\theta}}}\right\}. \tag{28}$$

$$\frac{\partial L}{\partial \alpha}=-n\psi\left(\alpha\right)-\sum_{i=1}^{n}\left(\frac{\beta}{x_i}\right)^{\theta}+n\psi\left(\alpha,\lambda\right). \tag{29}$$

$$\frac{\partial L}{\partial \lambda}=-n\psi\left(\lambda\right)+n\psi\left(\alpha,\lambda\right)+\sum_{i=1}^{n}\log\left\{1-e^{-\left(\frac{\beta}{x_i}\right)^{\theta}}\right\}. \tag{30}$$

The derivatives obtained in (27), (28), (29), and (30) are nonlinear equations and cannot be derived numerically. Statistical software such as the R program can be used to obtain the parameters of its estimates by applying the Newton-Raphson algorithm**.**

5. Application of the OBPF Distribution to Model Groundwater Quality Parameters

In this section, we fit the OBPF model to two groundwater quality data sets. The groundwater quality data sets used consist of physical (pH) and chemical (calcium) parameters. The performance of the OBPF distribution was compared to that of the alpha power transformed Fréchet (APTF) distribution by [45], the modified Fréchet (MF) distribution by [46], the generalized transmuted Fréchet (GTF) distribution by [47], and the gamma extended Fréchet (GF) distribution by [48]. The method of maximum likelihood is employed to estimate the parameters for the candidate models. The criteria for selecting the distribution with the best fit are log-likelihood, Bayesian Information Criteria (BIC), Akaike Information Criteria (AIC), Cramer–von Mises (CM), Anderson–Darling (AD), and Kolmogorov–Smirnov (KS) statistics (with p-values). In general, the smaller the values of these statistics, the better the fit to the data [49]. All the analyses in this study were performed using R software.

Data I: This data set represents the pH concentration of the groundwater obtained in the Jaen area, Kano State, Nigeria. The observations are as follows:

6.27, 6.14, 5.59, 6.7, 5.76, 6.97, 6.65, 6.5, 7.17, 6.97, 6.99, 6.9, 6.92, 6.82, 6.35, 7.12, 6.85, 6.71, 6.85, 7.22, 6.66, 7.07, 6.14, 6.41, 6.47, 6.32, 6.62, 7.2, 6.76, 7.04.

Table 1 shows the summary statistics of the pH data. From Table 1, the statistical summary reveals that the pH data exhibit skewness and kurtosis behaviors. In particular, negative skewness with low kurtosis coefficients for the pH concentration in the groundwater is observed. Therefore, this pH data is suitable for left-skewed statistical models such as the proposed OBPF model.
Figure 3 presents the histogram, kernel density, box, and violin plots of the pH data. From Figure 4, it is obvious that the pH data indicates statistically left-skewed behavior. Hence, this data is appropriate for skewed distributions. The shape of the total time on test (TTT) plot displayed in Figure 5 indicates that the pH data set has an increasing failure rate. This demonstrates that the proposed OBPF distribution's hazard function is appropriate for modelling this type of data.

Table 1 The summary statistics for the pH data.

Data	Min	Q1	Q3	Median	Mean	Max	Variance	Skewness	Kurtosis
pH	5.590	6.425	6.970	6.735	6.671	7.220	0.169	-0.826	0.054

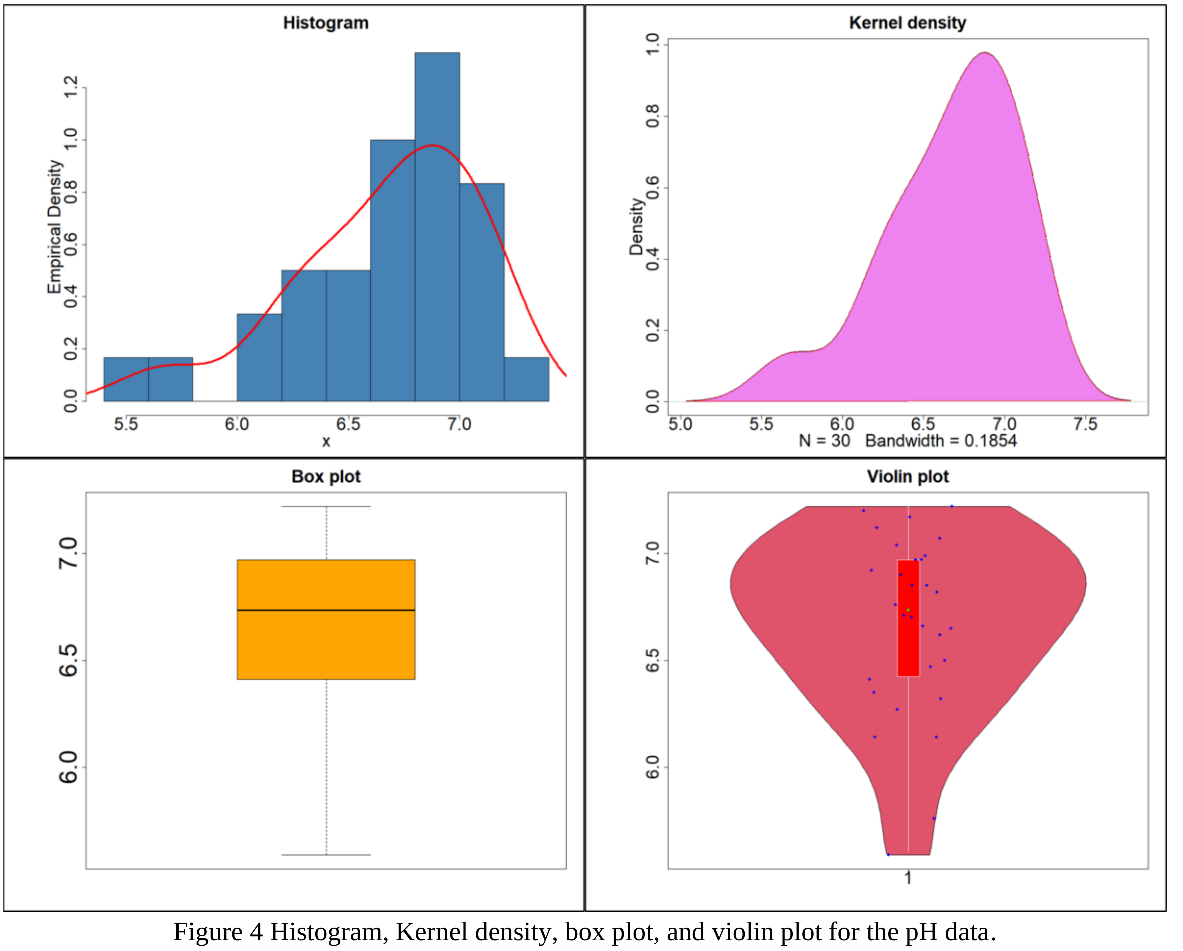

Figure 4 Histogram, Kernel density, box plot, and violin plot for the pH data.

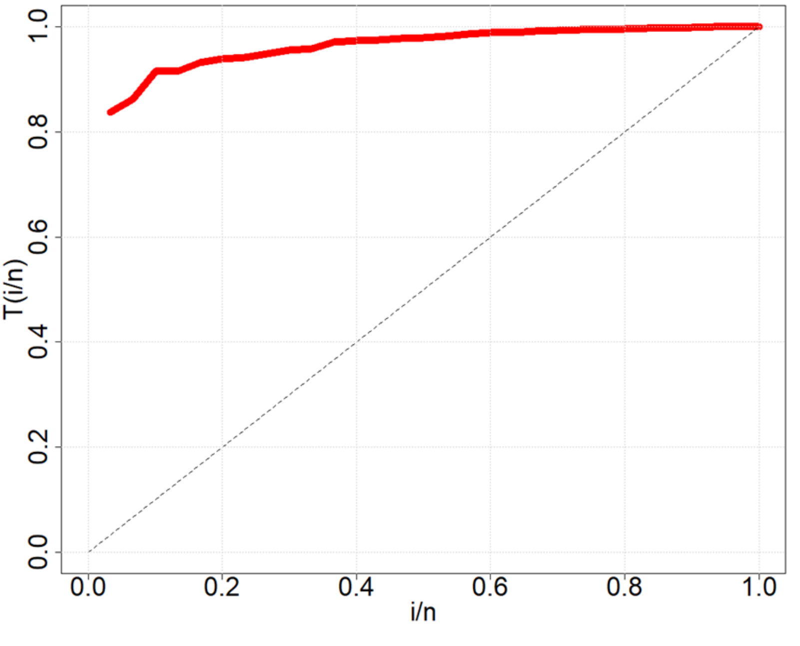

Figure 5 TTT plot for the pH data.

Table 2 shows the results of the MLEs and their corresponding standard errors (in parentheses) for the model parameter using the pH data. From Table 3, we conclude that the OBPF distribution provides the highest log-likelihood value, the lowest of the AIC, BIC, CM, AD, and KS statistics, and the largest p-value for the pH data. Therefore, the OBPF distribution is chosen as the model with the best fit among the distributions considered. Figure 6 shows the histogram and cdf plot of the OBPF, APTF, MF, GTF, and GF distributions concerning the pH concentrations. The OBPF distribution represents the curve with the magenta line, and it has been illustrated in Figure 6 to have the highest peak and provide a better fit to the histogram of the pH data.

Table 2 Estimates of the competitive models for the pH data.

Model	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$
OBPF	21.6453	6.8477	1.8456	0.3546
	(3.2155)	(0.0606)	(0.6724)	(0.0346)
APTF	6.7101	0.2276	1.6342	1.6421
	(0.0726)	(0.0345)	(0.2547)	(0.2473)
MF	6.6713	0.4045	0.7114	1.3453
	(0.0738)	(0.0522)	(0.1634)	(0.3421)
GTF	28.8659	6.7039	0.5864	1.3546
	(4.3966)	(0.0738)	(0.0154)	(0.5342)
GF	61.2296	19.1571	0.6753	0.2464
	(17.4082)	(3.1138)	(0.3428)	(0.0773)

Table 3 Statistics of the competitive models fitted to the pH data.

Model	$-\hat{\ell}$	**AIC**	**BIC**	**CM**	**AD**	**KS**	**p-value (KS)**
OBPF	12.8221	29.6442	32.4467	0.0260	0.1975	0.0693	0.8374
APTF	15.4036	34.8060	37.6084	0.0486	0.4287	0.0962	0.7453
MF	15.4167	34.8334	37.6358	0.0816	0.5541	0.1162	0.6453
GTF	15.9686	35.9373	38.7397	0.0558	0.5024	0.1052	0.4231
GF	15.9825	35.9650	38.7675	0.0950	0.6412	0.1253	0.3241

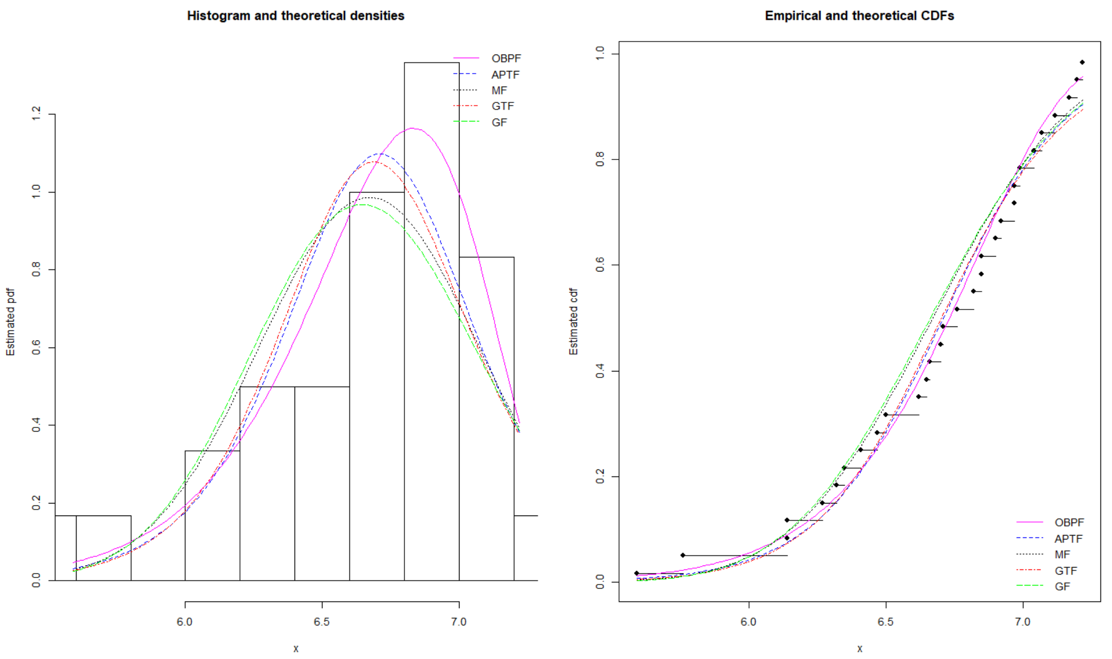

Figure 6 Estimated pdfs and cdfs plots of the OBPF and other competitive models for the pH data.

Data II: This data set corresponds to the calcium concentrations for the groundwater quality of the groundwater collected from the Jaen area, Kano State, Nigeria. The data set is given below:

1.75, 0.35, 0.87, 0.17, 0.29, 0.07, 0.69, 0.26, 0.26, 0.87, 0.28, 0.21, 1.44, 0.31, 0.36, 0.46, 0.27, 0.36, 0.24, 1.44, 0.21, 1.69, 0.28, 0.94, 1.32, 0.26, 0.33, 0.3, 1.1, 0.3.

Table 4 displays a statistical summary of calcium concentration. The table shows that the pH data is a right-skewed data due to the positive skewness coefficient, and a kurtosis value smaller than 3 suggests that the data has a platykurtic distribution.

The histogram, kernel density, box, and violin plots of this data are illustrated in Figure 7. The summary statistics and the empirical density plot for the calcium data reveal that the calcium data is positively skewed. This indicates that the OBPF distribution can be used to model data with right-skewness behavior. The pH data exhibits concave and convex failure rate, evident from the shape of the TTT plot depicted in Figure 8. This confirms that the shape of the hazard function of the proposed OBPF model given in Figure 3 is appropriate for modeling this type of data.

Table 4 The summary statistics for the calcium data.

Data	Min	Q1	Q3	Median	Mean	Max	Variance	Skewness	Kurtosis
Calcium	0.070	0.263	0.870	0.320	0.589	1.750	0.246	1.096	-0.269

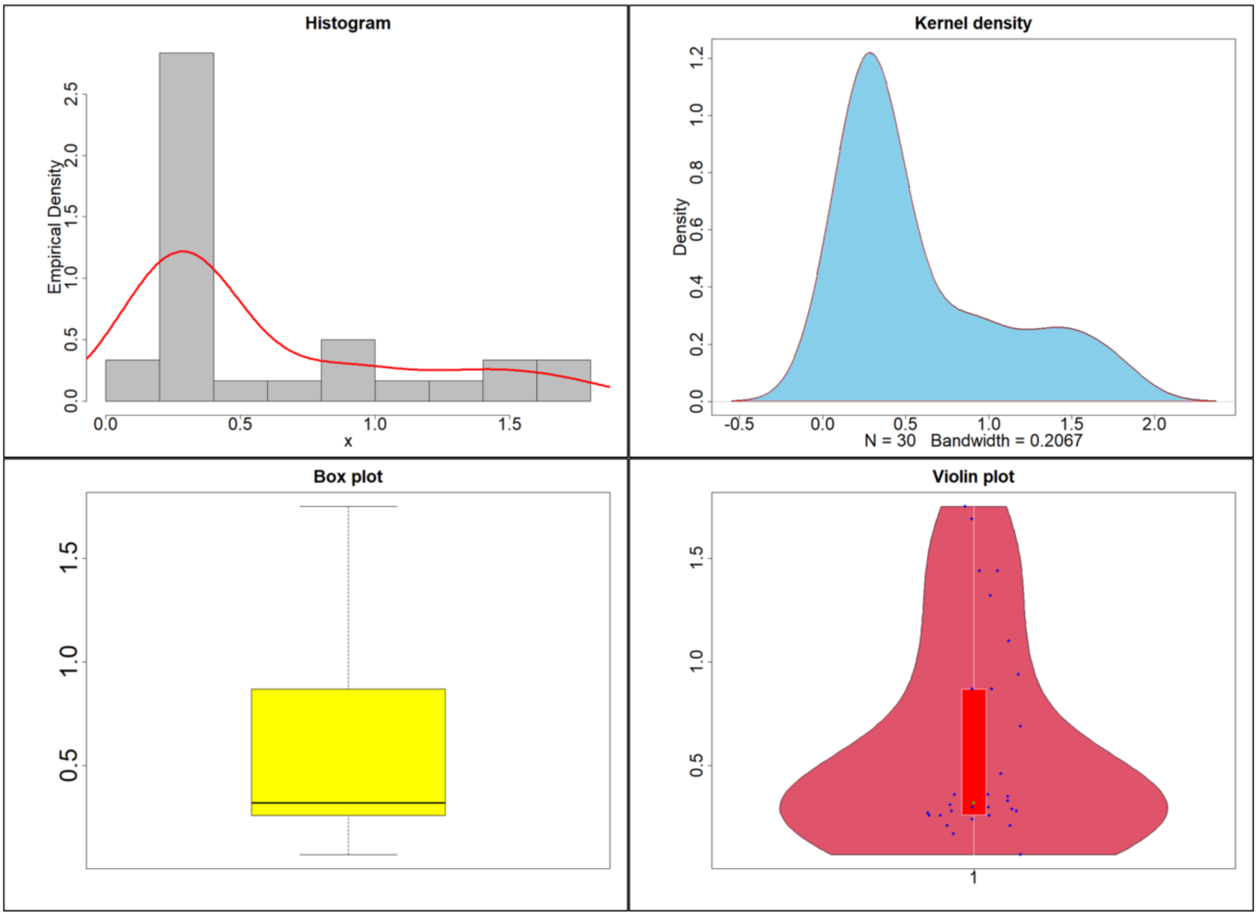

Figure 7 Histogram, Kernel density, box plot, and violin plot for the calcium data.

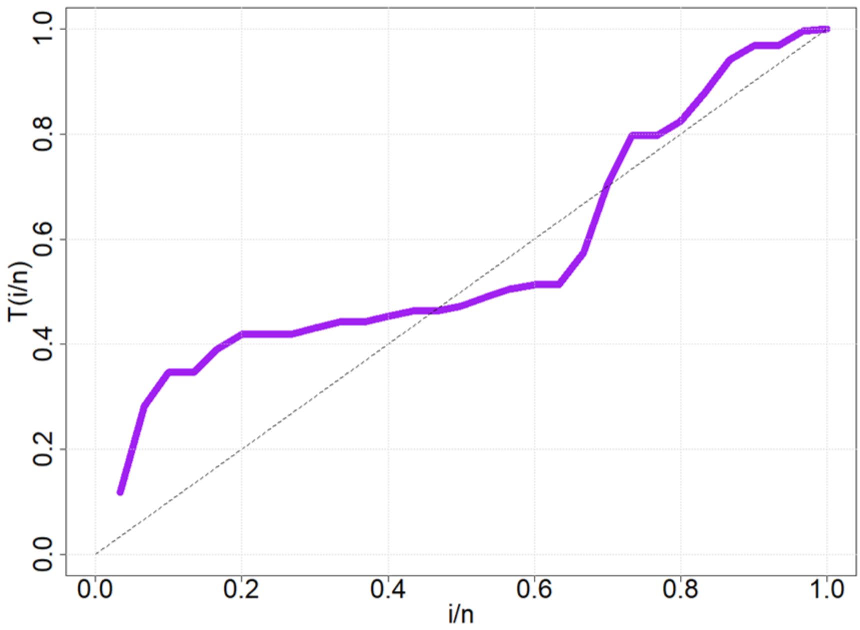

Figure 8 TTT plot for the calcium data.

Table 5 lists the MLEs and their corresponding standard errors (in parentheses) for the competitive model parameters. From the results in Table 6, we conclude that the OBPF model provides the best fit with the highest log-likelihood value, the lowest of the AIC, BIC, CM, AD, and KS statistics, and the largest p-value for the calcium data. Therefore, the proposed OBPF distribution is chosen as the model with the best fit among the distributions considered.

Figure 9 displays the histogram and cdf plots for the calcium data. The plots support the results in Table 6. Hence, the OBPF distribution could be the appropriate model for fitting the calcium concentration to the groundwater quality data.

Table 5 Estimates of the competitive models for the calcium data.

Model	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$
OBPF	0.8398	0.7810	0.3037	0.6453
	(0.1426)	(0.1008)	(0.0115)	(0.0367)
APTF	1.9154	0.6230	0.3054	1.3452
	(0.4580)	(0.1701)	(0.0354)	(0.1329)
MF	2.1989	0.4073	1.5463	1.5641
	(0.3330)	(0.0597)	(0.3354)	(0.2512)
GTF	1.7554	2.9786	1.6453	1.0542
	(0.4173)	(0.8184)	(0.2481)	(0.1324)
GF	1.3076	0.6443	1.2462	1.8538
	(0.1792)	(0.0955)	(0.6342)	(0.7432)

Table 6 Statistics of the competitive models fitted to the calcium data.

Model	$-\hat{\ell}$	**AIC**	**BIC**	**CM**	**AD**	**KS**	**p-value (KS)**
OBPF	9.9593	23.9188	26.7212	0.2099	1.3549	0.2254	0.7934
APTF	11.7252	27.4505	30.2528	0.3498	1.8209	0.2668	0.3658
MF	12.4618	28.9237	31.7261	0.3551	1.7718	0.2601	0.2429
GTF	10.1920	24.3841	27.1866	0.2246	1.3902	0.2459	0.7241
GF	10.6297	25.2594	28.0618	0.2379	1.4560	0.2309	0.6574

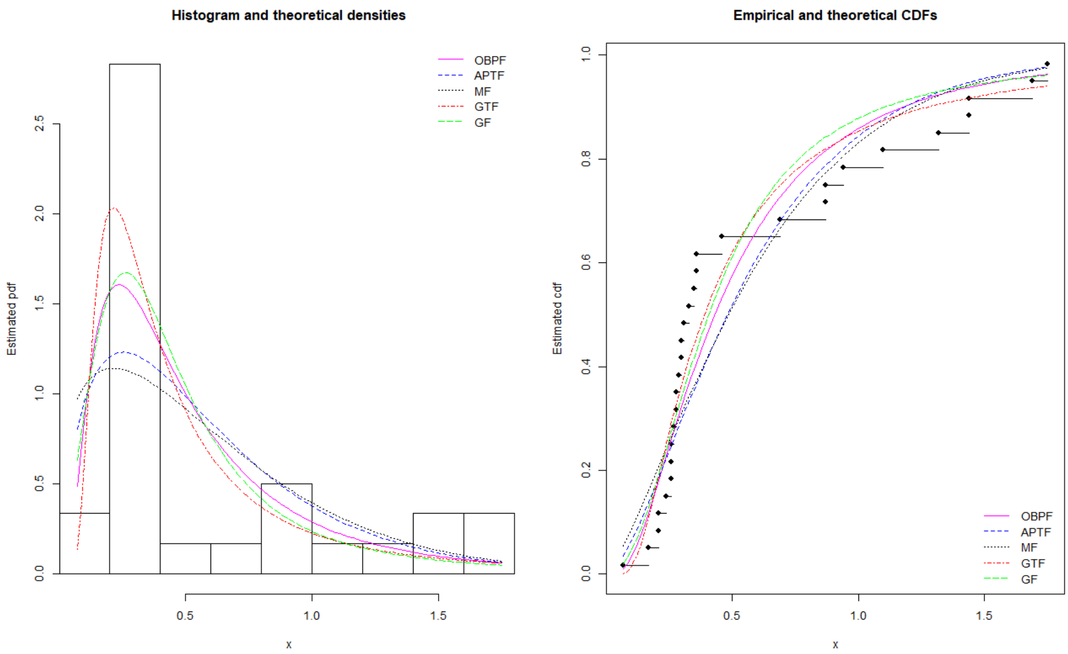

Figure 9 Estimated pdfs and cdfs plots of the OBPF and other competitive models for the calcium data.

6. Conclusion

In this research, we proposed a novel extension of the Fréchet distribution by introducing two extra shape parameters, thus defining the odd beta prime Fréchet (OBPF). The probability density and hazard function of the new distribution show remarkable flexibility. Various statistical features for the proposed distribution are obtained, such as mixture representation for the density function, moments, incomplete moments, order statistics, and quantile function. The maximum likelihood method is utilized for estimating the model parameters. An application to two groundwater quality data sets shows that the fit of the new model is superior to the fits of the well-established extended Fréchet distributions. The results indicate that the proposed OBPF model is suitable to fit data with right- and left-skewed behaviors. We hope that the proposed model may attract wider applications in statistics.

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References

[1] H. A. Loáiciga, “Probability Distributions in Groundwater Hydrology: Methods and Applications,” *J. Hydrol. Eng.*, vol. 20, no. 5, p. 4014063, doi: 10.1061/(ASCE)HE.1943-5584.0001061.

[2] A. A. Suleiman, U. A. Abdullahi, A. Suleiman, R. B. Yunus, and S. A. Suleiman, “Assessment of Groundwater Quality Using Multivariate Statistical Techniques,” in *Intelligent Systems Modeling and Simulation II: Machine Learning, Neural Networks, Efficient Numerical Algorithm and Statistical*, Methods: Springer, pp. 567–579.

[3] A. A. Suleiman, A. Ibrahim, and U. A. Abdullahi, “Statistical explanatory assessment of groundwater quality in Gwale LGA, Kano state, northwest Nigeria,” *Hydrospatial Anal.*, vol. 4, no. 1, pp. 1–13,.

[4] V. V Singh, A. A. Suleman, A. Ibrahim, U. A. Abdullahi, and S. A. Suleiman, “Assessment of probability distributions of groundwater quality data in Gwale area, north-western Nigeria,” *Ann. Optim. Theory Pract.*, vol. 3, no. 1, pp. 37–46,.

[5] M. Fréchet, “Sur la loi des erreurs d’observation,” *Математический сборник*, vol. 32, no. 1, pp. 5–8,.

[6] S. Kotz and S. Nadarajah, “Extreme value distributions: theory and applications,” *world Sci.*

[7] H. M. Alshanbari, A. M. Gemeay, A. A.-A. H. El-Bagoury, S. K. Khosa, E. H. Hafez, and A. H. Muse, “A novel extension of Fréchet distribution: Application on real data and simulation,” *Alexandria Eng. J.*, vol. 61, no. 10, pp. 7917–7938, doi: 10.1016/j.aej.2022.01.013.

[8] F. C. Morales, *Estimation of max-stable processes using Monte Carlo methods with applications to financial risk assessment*. The University of North Carolina at Chapel Hill.

[9] A. Zaharim, S. K. Najid, A. M. Razali, and K. Sopian, “Analyzing Malaysian wind speed data using statistical distribution,” in *Proceedings of the 4th IASME/WSEAS International conference on energy and environment*, vol. 2426, p. 360370.

[10] M. Mubarak, “Parameter estimation based on the Fréchet progressive type II censored data with binomial

removals," *J. Qual. Reliab. Eng.*, vol. 2012.
[11] M. Eliwa, "A new one-parameter lifetime distribution and its regression model with applications," *PLoS One*, vol. 16, no. 2, p. 246969.
[12] S. Nasiru, "Extended Odd Fréchet-G Family of Distributions," *J. Probab. Stat.*, vol. 2018, p. 2931326, doi: 10.1155/2018/2931326.
[13] M. El-Morshedy, F. S. Alshammari, Y. S. Hamed, M. S. Eliwa, and H. M. Yousof, "A New Family of Continuous Probability Distributions," *Entropy*, vol. 23, no. 2, p. 194, [Online]. Available: https://www.mdpi.com/1099-4300/23/2/194.
[14] A. M. Gemeay, "General two-parameter distribution: Statistical properties, estimation, and application on COVID-19," *PLoS One*, vol. 18, no. 2, p. 281474, doi: 10.1371/journal.pone.0281474.
[15] H. M. Yousof, E. Altun, T. G. Ramires, M. Alizadeh, and M. Rasekhi, "A new family of distributions with properties, regression models and applications," *J. Stat. Manag. Syst.*, vol. 21, no. 1, pp. 163–188, doi: 10.1080/09720510.2017.1411028.
[16] S. Nadarajah and A. Gupta, "The beta Fréchet distribution," *Far east J. Theor. Stat.*, vol. 14, no. 1, pp. 15–24,.
[17] A. Z. Afify, H. M. Yousof, G. M. Cordeiro, E. M. M. Ortega, and Z. M. Nofal, "The Weibull Fréchet distribution and its applications," *J. Appl. Stat.*, vol. 43, no. 14, pp. 2608–2626, 2016, doi: 10.1080/02664763.2016.1142945.
[18] M. Hamed, F. Aldossary, and A. Z. Afify, "The four-parameter Fréchet distribution: Properties and applications," *Pakistan J. Stat. Oper. Res.*, pp. 249–264,.
[19] M. M. Mansour, E. Elrazik, E. Altun, A. Z. Afify, and Z. Iqbal, "Pak," *j. Stat.*, vol. 34, no. 6, pp. 441–458.
[20] P. E. Oguntunde, M. A. Khaleel, M. T. Ahmed, and H. I. Okagbue, "The Gompertz fréchet distribution: properties and applications," *Cogent Math. Stat.*, vol. 6, no. 1, p. 1568662.
[21] R. H. A. El Khaleq, "The Generalized Odd Log-Logistic Fréchet Distribution for Modeling Extreme Values," *Pakistan J. Stat. Oper. Res.*, pp. 649–674,.
[22] S. E. A. Rasool and S. F. Mohammed, "New odd Chen Fréchet distributions: Properties and applications," *Int. J. Nonlinear Anal. Appl.*
[23] N. Alsadat, "The novel Kumaraswamy power Frechet distribution with data analysis related to diverse scientific areas," *Alexandria Eng. J.*, vol. 70, pp. 651–664, doi: 10.1016/j.aej.2023.03.003.
[24] A. A. Suleiman, H. Daud, N. S. S. Singh, M. Othman, A. I. Ishaq, and R. Sokkalingam, "A Novel Odd Beta Prime-Logistic Distribution: Desirable Mathematical Properties and Applications to Engineering and Environmental Data," *Sustainability*, vol. 15, no. 13, p. 10239, 2023.
[25] A. A. Suleiman, H. Daud, N. S. S. Singh, M. Othman, A. I. Ishaq, and R. Sokkalingam,"The Odd Beta Prime-G Family of Probability Distributions: Properties and Applications to Engineering and Environmental Data," in Computer Sciences & Mathematics Forum, 2023, vol. 7, no. 1: MDPI, p. 20.
[26] A. A. Suleiman, "A New Statistical Model Based on the Novel Generalized Odd Beta Prime Family of Continuous Probability Distributions with Applications to Cancer Disease Data Sets." .
[27] G. M. Cordeiro, M. Alizadeh, T. G. Ramires, and E. M. M. Ortega, "The generalized odd half-Cauchy family of distributions: Properties and applications," *Commun. Stat. - Theory Methods*, vol. 46, no. 11, pp. 5685–5705, doi: 10.1080/03610926.2015.1109665.
[28] M. Alizadeh, G. M. Cordeiro, A. D. C. Nascimento, M. C. S. Lima, and E. M. M. Ortega, "Odd-Burr generalized family of distributions with some applications," *J. Stat. Comput. Simul.*, vol. 87, no. 2, pp. 367–389, doi: 10.1080/00949655.2016.1209200.
[29] Ã. Mustafa, M. Alizadeh, H. M. Yousof, and N. S. Butt, "The generalized odd Weibull generated family of distributions: statistical properties and applications," *Pakistan J. Stat. Oper. Res.*, pp. 541–556,.
[30] B. Hosseini, M. Afshari, and M. Alizadeh, "The generalized odd gamma-g family of distributions: properties and applications," *Austrian J. Stat.*, vol. 47, no. 2, pp. 69–89,.
[32] C. Chesneau and T. E. Achi, "Modified Odd Weibull Family of Distributions: Properties and Applications," *J. Indian Soc. Probab. Stat.*, vol. 21, no. 1, pp. 259–286, doi: 10.1007/s41096-020-00075-x.
[32] A. Z. Afify and M. Alizadeh, "The odd Dagum family of distributions: Properties and applications," *J. Appl. Probab. Stat.*, vol. 15, no. 1, pp. 45–72,.
[33] Z. Ahmad, "The Zubair-G Family of Distributions: Properties and Applications," *Ann. Data Sci.*, vol. 7, no. 2, pp. 195–208, doi: 10.1007/s40745-018-0169-9.
[34] R. A. R. Bantan, C. Chesneau, F. Jamal, I. Elbatal, and M. Elgarhy, "The Truncated Burr X-G Family of Distributions: Properties and Applications to Actuarial and Financial Data," *Entropy*, vol. 23, no. 8, p. 1088, [Online]. Available: https://www.mdpi.com/1099-4300/23/8/1088.
[35] J. T. Eghwerido, P. E. Oguntunde, and F. I. Agu, "The Alpha Power Marshall-Olkin-G Distribution: Properties, and Applications," *Sankhya A*, doi: 10.1007/s13171-020-00235-y.
[36] A. I. Ishaq and A. A. Abiodun, "The Maxwell–Weibull distribution in modeling lifetime datasets," *Ann. Data Sci.*, vol. 7, no. 4, pp. 639–662,.
[37] R. Bantan, F. Jamal, C. Chesneau, and M. Elgarhy, "Type II Power Topp-Leone Generated Family of

Distributions with Statistical Inference and Applications," *Symmetry (Basel).*, vol. 12, no. 1.
[38] K. Hamid, A. Mahmoud, A. Morad, and M. Y. Haitham, "The Odd Log-Logistic Burr-X Family of Distributions: Properties and Applications," *J. Stat. Theory Appl.*, vol. 20, no. 2, pp. 228–241, doi: 10.2991/jsta.d.210609.001.
[39] N. Alotaibi, I. Elbatal, E. M. Almetwally, S. A. Alyami, A. Al-Moisheer, and M. Elgarhy, "Truncated Cauchy power Weibull-G class of distributions: Bayesian and non-Bayesian inference modelling for COVID-19 and carbon fiber data," *Mathematics*, vol. 10, no. 9, p. 1565.
[40] S. A. Alyami, M. G. Babu, I. Elbatal, N. Alotaibi, and M. Elgarhy, "Type II Half-Logistic Odd Fréchet Class of Distributions: Statistical Theory and Applications," *Symmetry (Basel).*, vol. 14, no. 6, p. 1222.
[41] M. M. Badr, I. Elbatal, F. Jamal, C. Chesneau, and M. Elgarhy, "The transmuted odd Fréchet-G family of distributions: Theory and applications," *Mathematics*, vol. 8, no. 6, p. 958.
[42] I. Elbatal, N. Alotaibi, E. M. Almetwally, S. A. Alyami, and M. Elgarhy, "On odd perks-G class of distributions: properties, regression model, discretization, Bayesian and non-Bayesian estimation, and applications," *Symmetry (Basel).*, vol. 14, no. 5, p. 883.
[43] A. M. Almarashi, F. Jamal, C. Chesneau, and M. Elgarhy, "A new truncated muth generated family of distributions with applications," *Complexity*, vol. 2021, pp. 1–14,.
[44] A. Ishaq, A. Usman, M. Tasi'u, A. Suleiman, and A. Ahmad, "A New Odd F-Weibull Distribution: Properties and Application of the Monthly Nigerian Naira to British Pound Exchange Rate Data," in *2022 International Conference on Data Analytics for Business and Industry (ICDABI)*, 2022: IEEE, pp. 326-332.
[45] S. Nasiru, P. N. Mwita, and O. Ngesa, "Alpha power transformed Frechet distribution," *Appl. Math. Inf. Sci.*, vol. 13, no. 1, pp. 129–141,.
[46] C. J. Tablada and G. M. Cordeiro, "The modified Fréchet distribution and its properties," *Commun. Stat. - Theory Methods*, vol. 46, no. 21, pp. 10617–10639, doi: 10.1080/03610926.2016.1239115.
[47] Z. M. Nofal and M. Ahsanullah, "A new extension of the Fréchet distribution: Properties and its characterization," *Commun. Stat. - Theory Methods*, vol. 48, no. 9, pp. 2267–2285, doi: 10.1080/03610926.2018.1465080.
[48] R. V Silva, T. A. Andrade, D. B. Maciel, R. P. Campos, and G. M. Cordeiro, "A New Lifetime Model: The Gamma Extended Fréchet Distribution," *J. Stat. Theory Appl*, vol. 12, no. 1, pp. 39–54,.
[49] U. A. Abdullahi, A. A. Suleiman, A. I. Ishaq, A. Usman, and A. Suleiman, "The Maxwell Exponential Distribution: Theory and Application to Lifetime Data," *J. Stat. Model. Anal.*, vol. 3, no. 2.